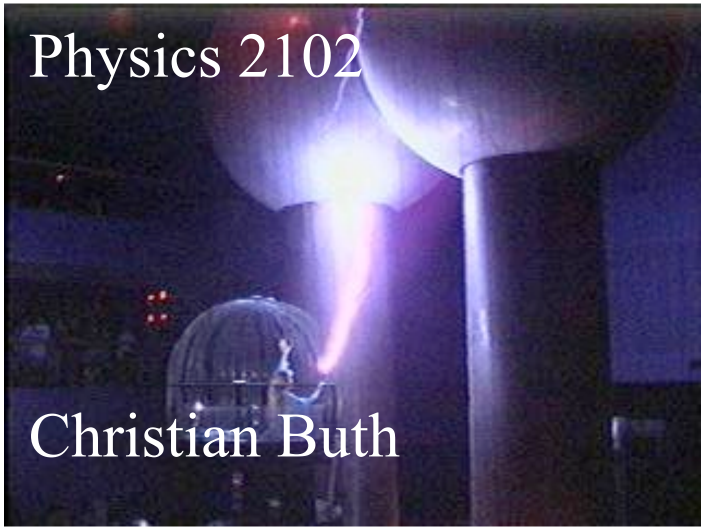
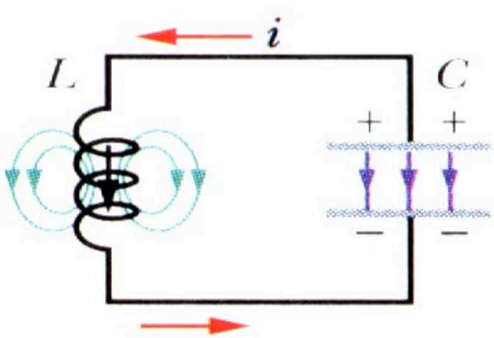


Lecture 29

Electrical Oscillations, LC Circuits

03/25/2009



Review

• Notion of electric potential **does not work** for electric fields produced by induction

• **Inductance** of a solenoid

$$\Phi_B = NAB = Li$$

• SI unit **henry**

• **Self induction**: an EMF appears in any coil in which the current is changing: $EMF = -L\frac{di}{dt}$

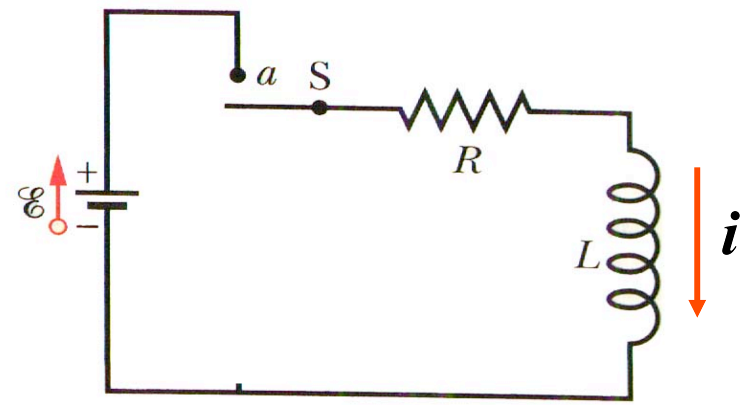
• **Direction** of self-induced EMF from Lenz's law

• "Charging" an inductor: $i = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$

• "Discharging" an inductor: $i = \frac{\mathcal{E}}{R}e^{-\frac{Rt}{L}}$

Inductors & Energy

- Recall that **capacitors** store energy in an **electric** field
- **Inductors** store energy in a **magnetic** field



$$\mathcal{E} = iR + L\frac{di}{dt}$$

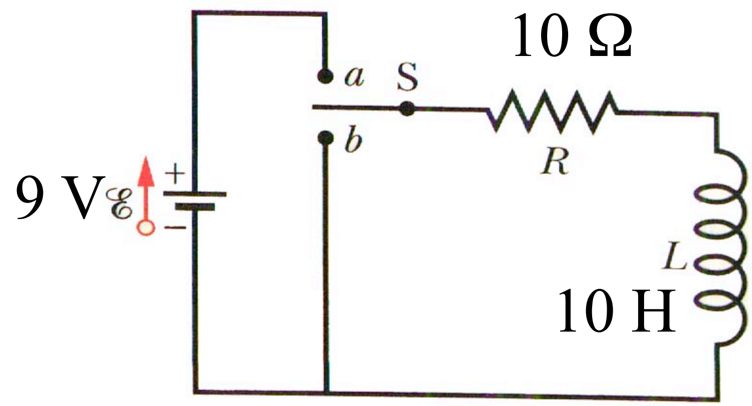
$$(i\mathcal{E}) = (i^2R) + Li\frac{di}{dt} \quad \Rightarrow \quad (i\mathcal{E}) = (i^2R) + \frac{d}{dt}\left(\frac{Li^2}{2}\right)$$

Power delivered by battery = power dissipated by R

+ (d/dt) energy stored in L

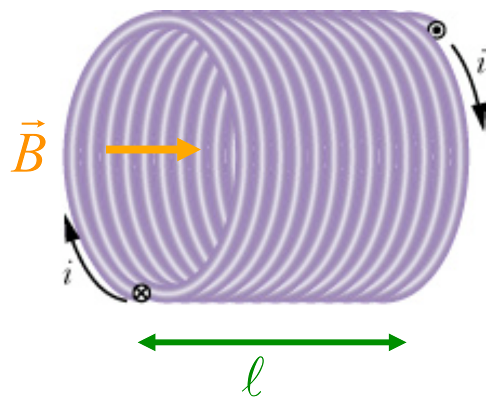
Example

- The switch has been in position "a" for a long time.
- It is now moved to position "b" without breaking the circuit.
- What is the total energy dissipated by the resistor until the circuit reaches equilibrium?



- When switch has been in position "a" for long time, current through inductor = (9V)/(10Ω) = 0.9A.
- Energy stored in inductor = (0.5)(10H)$(0.9\text{A})^2$ = 4.05 J
- When inductor "discharges" through the resistor, all this stored energy is dissipated as heat = 4.05 J.

Energy Density of a Magnetic Field



- **Energy stored** in solenoid

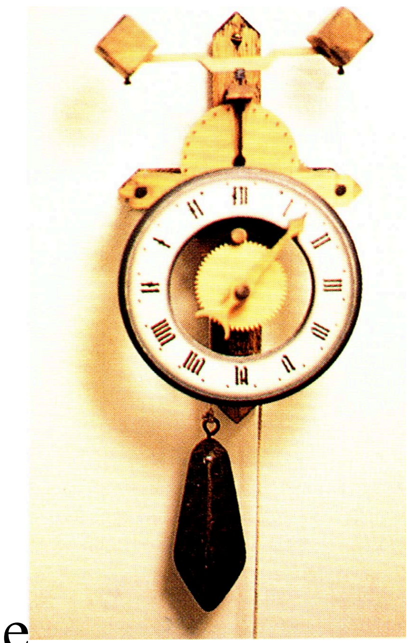
$$U_B = \frac{1}{2}Li^2 = \frac{\mu_0 n^2 A l i^2}{2} \qquad L = \mu_0 n^2 A l$$

- **Energy density** $u_B = \frac{U_B}{V} \qquad B = \mu_0 i n$

$$u_B = \frac{\mu_0 n^2 i^2}{2} = \frac{B^2}{2\mu_0}$$

- Although derived for a special case, expression **holds generally**

Oscillators in Physics

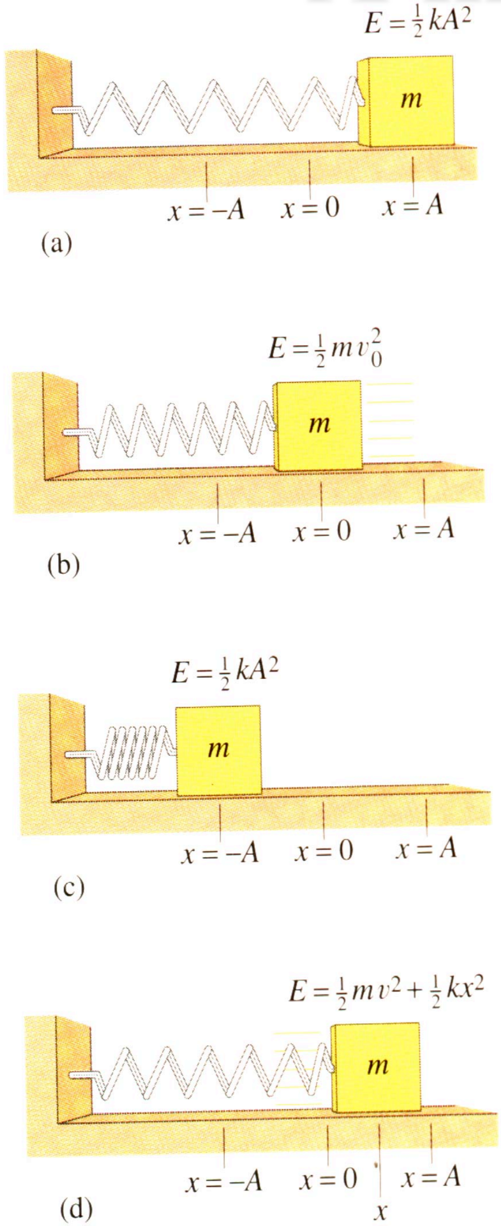


Oscillators are very useful in practical applications, for instance, to keep time, or to focus energy in a system.

All oscillators operate along the same principle: they are systems that can store energy in more than one way and exchange it back and forth between the different storage possibilities. For instance, in pendulums (and swings) one **exchanges energy** between **kinetic and potential** form.

In this course we have studied that **coils and capacitors** are devices that can store **electromagnetic energy**. In one case it is stored in a magnetic field, in the other in an electric field.

A mechanical oscillator



$$E_{tot} = E_{kin} + E_{pot} \qquad E_{tot} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

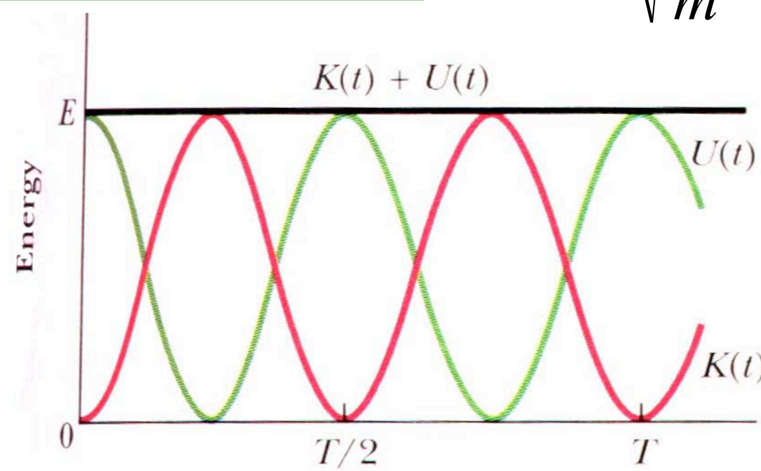
$$\frac{dE_{tot}}{dt} = 0 = \frac{1}{2} m \left(2v \frac{dv}{dt} \right) + \frac{1}{2} k \left(2x \frac{dx}{dt} \right) \qquad v = \frac{dx}{dt}$$

$$m \frac{dv}{dt} + k x = 0 \qquad m \frac{d^2 x}{dt^2} + k x = 0$$

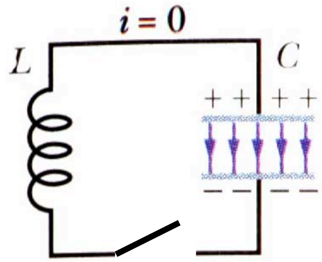
Newton's law
F=ma!

$$\text{Solution:} \quad x(t) = x_0 \cos(\omega t + \phi_0) \qquad \omega = \sqrt{\frac{k}{m}}$$

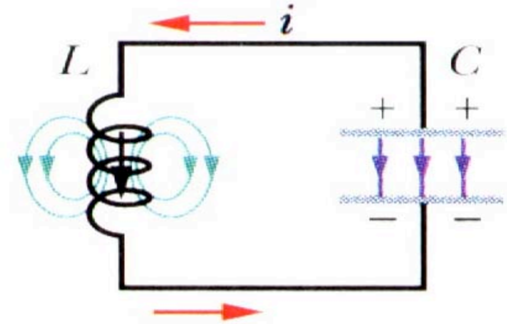
x_0 : amplitude
ω : frequency
ϕ_0 : phase



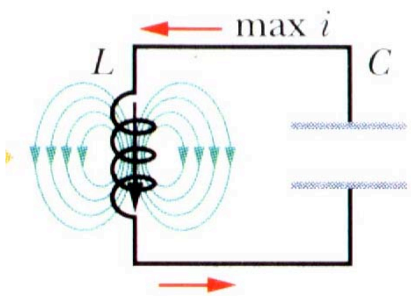
An electromagnetic oscillator



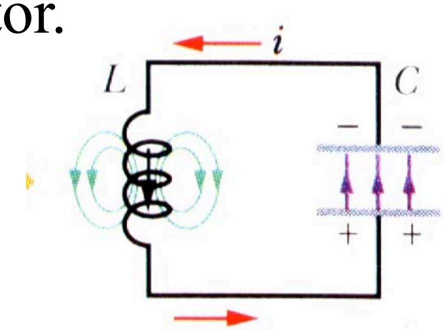
Capacitor initially charged. Initially, current is zero, energy is all stored in the capacitor.



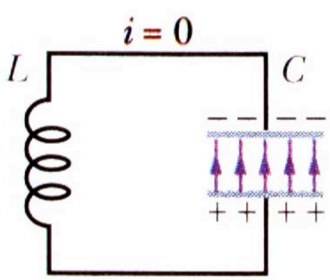
A current gets going, energy gets split between the capacitor and the inductor.



Capacitor discharges completely, yet current keeps going. Energy is all in the inductor.

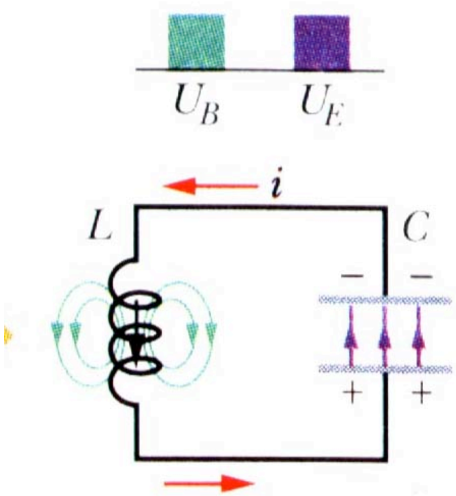


The magnetic field on the coil starts to collapse, which will start to recharge the capacitor.



Finally, we reach the same state we started with (with opposite polarity) and the cycle restarts.

Electric Oscillators: the math



$$E_{tot} = E_{mag} + E_{elec} \qquad E_{tot} = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}$$

$$\frac{dE_{tot}}{dt} = 0 = \frac{1}{2}L\left(2i\frac{di}{dt}\right) + \frac{1}{2C}\left(2q\frac{dq}{dt}\right) \qquad i = \frac{dq}{dt}$$

$$0 = L\left(\frac{di}{dt}\right) + \frac{1}{C}(q) \quad \text{(the loop rule!)}$$

$$\boxed{0 = L\frac{d^2q}{dt^2} + \frac{q}{C}}$$

Compare with: $\boxed{m\frac{d^2x}{dt^2} + kx = 0}$

Analogy between electrical and mechanical oscillations:

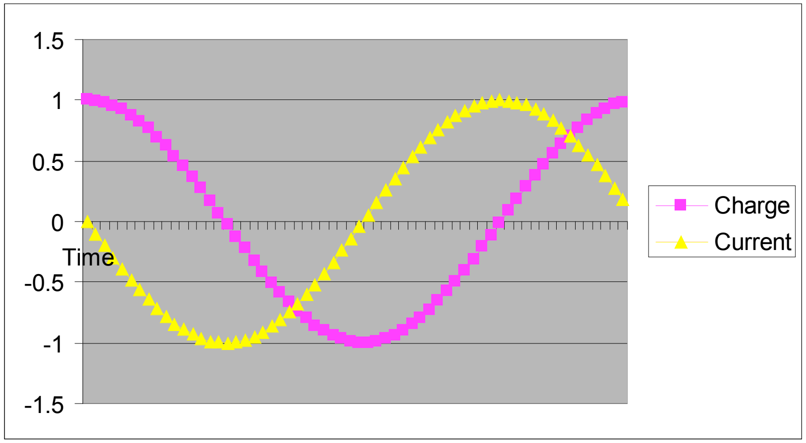
$q \rightarrow x \qquad 1/C \rightarrow k$

$i \rightarrow v \qquad L \rightarrow M$

$$x(t) = x_0\cos(\omega t + \phi_0) \qquad \boxed{q = q_0\cos(\omega t + \phi_0)}$$

$$\omega = \sqrt{\frac{k}{m}} \qquad \boxed{\omega = \sqrt{\frac{1}{LC}}}$$

Electric Oscillators: the math

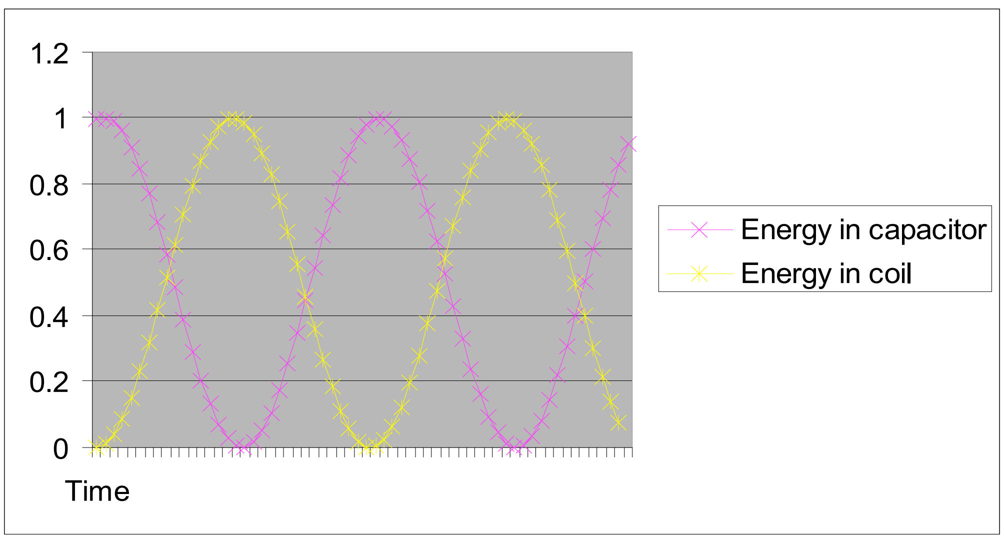


$$q = q_0 \cos(\omega t + \phi_0)$$

$$i = \frac{dq}{dt} = -\omega q_0 \sin(\omega t + \phi_0)$$

$$E_{mag} = \frac{1}{2} L i^2 = \frac{1}{2} L \omega^2 q_0^{\ 2} \sin^2(\omega t + \phi_0)$$

$$E_{ele} = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2C} q_0^{\ 2} \cos^2(\omega t + \phi_0)$$



And remembering that,

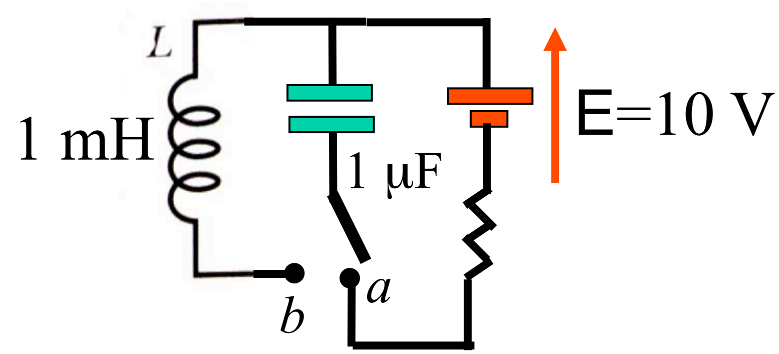
$$\cos^2 x + \sin^2 x = 1, \text{and } \omega = \sqrt{\frac{1}{LC}}$$

$$E_{tot} = E_{mag} + E_{ele} = \frac{1}{2C} q_0^{\ 2}$$

The energy is constant and equal to what we started with.

Example

- In the circuit shown, the switch is in position "a" for a long time. It is then thrown to position "b."
- Calculate the amplitude of the resulting oscillating current.



$$q = q_0 \cos(\omega t + \phi_0)$$

$$i = \frac{dq}{dt} = -\omega q_0 \sin(\omega t + \phi_0)$$

- Switch in position "a": charge on capacitor: (1 μF)(10 V) = 10 μC
- Switch in position "b": maximum charge on capacitor: q_0 = 10 μC
- So, amplitude of oscillating current =

$$\omega q_0 = \frac{1}{\sqrt{(1mH)(1\mu F)}}(10\mu C) = \quad 0.316 \text{ A}$$

Summary 1

- **Energy** stored in a magnetic field

$$U_B = \frac{Li^2}{2}$$

- **Energy density** in magnetic field

$$u_B = \frac{B^2}{2\mu_0}$$

Summary 2

- Combining a capacitor and an inductor produces an **electrical oscillator**
- Total energy in circuit is **conserved**: switches between capacitor (electric field) and inductor (magnetic field)
- Differential equation of **LC circuit**:

$$0 = L\frac{d^2q}{dt^2} + \frac{q}{C}$$

$$q = q_0 \cos(\omega t + \phi_0)$$

- **Natural frequency** of oscillator is $\omega = 1/\sqrt{LC}$